Practice Calculations

$$V = (x-2y+z,\ 3yz^2,\ 2x+3y^2)$$

① curl $V = \nabla \times V =$

② div $V = \nabla \cdot V =$ (guess what this is)

$$\alpha = 2xy\,dx - 3y\,dz + x\,dy$$

③ $d\alpha =$

$$\beta = 2xy\,dx\wedge dy - 3\,dy\wedge dz + 2zy\,dx\wedge dz$$

④ $d\beta =$

Announcements: ① Hmk moved & changed (3.32)
② Test 2 corrections due 4/23
③ Test 3 corrections due 5/1

$$V = (x-2y+z,\ 3yz^2,\ 2x+3y^2)$$

① curl $V = \nabla \times V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x-2y+z & 3yz^2 & 2x+3y^2 \end{vmatrix} = i\left(\partial_y(2x+3y^2) - \partial_z(3yz^2)\right)$$

$$-j\left(\partial_x(2x+3y^2) - \partial_z(x-2y+z)\right) + k\left(\partial_x(3yz^2) - \partial_y(x-2y+z)\right)$$

$$= i(6y - 6yz) - j(2-1) + k(0-(-2))$$

$$\text{curl } V = ((6y - 6yz), -1, 2)$$

↑ tells us how much V is "curling" at (x,y,z) + direction of curl axis (Right hand Rule)

Curl V = directed inside the paper.

curl W = out toward us.

"measures circulation"

$$V = (x - 2y + z,\ 3yz^2,\ 2x + 3y^2)$$

② div $V = \nabla \cdot V = (\partial_x, \partial_y, \partial_z) \cdot (V_1, V_2, V_3)$

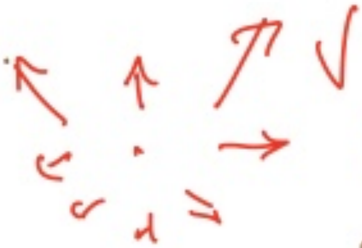
"divergence"

$$= \partial_x V_1 + \partial_y V_2 + \partial_z V_3$$

$$= \partial_x(x - 2y + z) + \partial_y(3yz^2) + \partial_z(2x + 3y^2)$$

$$= 1 + 3z^2 + 0 = 1 + 3z^2$$

↳ measures at each pt if volumes are increasing/decreasing.



div V > 0

div W < 0

volume looks like it is lengthing → positive divergence.

looks like zero divergence (objects would turn but not change volume)

(But curl would be towards us.)

$$\alpha = 2xy\,dx - 3y\,dz + x\,dy$$

(3) $$d\alpha = d(2xy)\wedge dx - d(3y)\wedge dz + d(x)\wedge dy$$

$$= (2y\,dx + 2x\,dy)\wedge dx - (3dy)\wedge dz + dx\wedge dy$$

$$\Rightarrow d\alpha = 2x\,\underbrace{dy\wedge dx}_{(-dx\wedge dy)} - 3\,dy\wedge dz + dx\wedge dy$$

$$\Rightarrow d\alpha = (1-2x)\,dx\wedge dy - 3\,dy\wedge dz$$

Note: $\alpha = 2xy\,dx - 3y\,dz + x\,dy = V \cdot ds$

where V is the vector field

$$V = (2xy, x, -3y)$$

$$\oint V \cdot ds = \int (2xy\,dx - 3y\,dz + x\,dy).$$

$$\operatorname{curl} V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & x & -3y \end{vmatrix}$$

$$= i\left(\partial_y(-3y) - \partial_z(x)\right) - j\left(\partial_x(-3y) - \partial_z(2xy)\right) + k\left(\partial_x(x) - \partial_y(2xy)\right)$$

$$= i(-3-0) - j(0-0) + k(1-2x)$$

$$= (-3, 0, 1-2x)$$

$$\boxed{\operatorname{Curl}(2xy, x, -3y) = (-3, 0, 1-2x)}$$

Also we got

$$d(2xy\,dx + x\,dy - 3y\,dz) = -3\underbrace{dy\wedge dz}_{x \text{ direction}} + 0 + (1-2x)\underbrace{dx\wedge dy}_{z \text{ direction}}$$

x direction $\longleftrightarrow dy \wedge dz \longleftrightarrow dx$

y direction $\longleftrightarrow dz \wedge dx \longleftrightarrow dy$

z direction $\longleftrightarrow dx \wedge dy \longleftrightarrow dz$

How do I know $dy \leftrightarrow$ y direction $\Leftrightarrow dz \wedge dx$ instead of $dx \wedge dz$?

Reason: $dy \wedge (dx \wedge dz) = -dx \wedge dy \wedge dz$
negative volume form

☆ $dy \wedge (dz \wedge dx) = -dy \wedge dx \wedge dz$
right one.
$= +dx \wedge dy \wedge dz$
positive volume form.

Using this recipe, we can view differential form integrals as vector field integrals and vice versa.

The Vector field V is conservative

$\Longleftrightarrow \text{curl}\, V = 0$ ← works in 3-d (also 2-d

$\Longleftrightarrow d(V^\flat) = 0$ ← works in any dimensions.

$V^\flat$ ← differential 1-form associated to vector field V.

Big Theorem - Stoke's Theorem

$$\int_{\partial R} \alpha = \int_R d\alpha,$$

where α is a differential form
(1-form, 2-form, ...)

R is a region (2-d, 3-d)

∂R = boundary of R ← orientation induced from orientation of R.

<u>Example</u> α is a function (0-form)

$\alpha = f$

$$\int_{\partial(\text{interval})} f = \int_{\text{interval}} df$$

$$= \int_a^b \underbrace{f'(x)\,dx}_{df} \overset{FTC}{=} \underbrace{f(b) - f(a)}_{\substack{\text{integral of } f \\ \text{over 2 points} \\ \text{(oriented)}}}$$

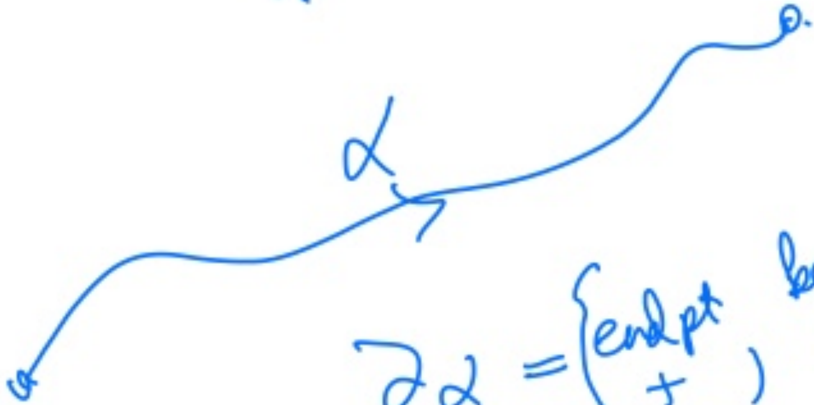
boundary of $[a,b]$ is $\{\underset{-}{a}, \underset{+}{b}\}$ two points.

Higher dimensions

If V is conservative, $V = \nabla f$

$V^\flat = df$

$$\oint_\alpha V \cdot ds = f\binom{\text{end}}{\text{pt}} - f\binom{\text{beg}}{\text{pt}}$$

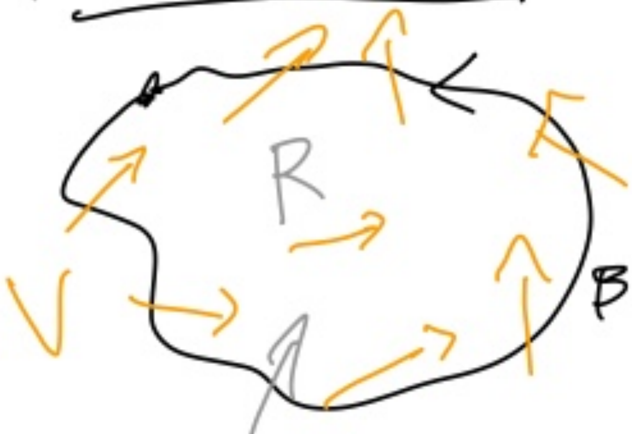


$$\partial\alpha = \{\underset{+}{\text{end pt}}, \underset{-}{\text{beg pt}}\}$$

<u>New calculation:</u>

$$\oint_\beta V \cdot ds = \int_\beta V^\flat$$

β ↖ loop



$\beta = \partial R$

$$= \int_{\partial R} V^\flat$$

$$\overset{\text{stokes}}{=} \int_R \underbrace{dV^\flat}_{(\text{curl} V) \cdot dA}$$

"Green's Thm" $\mathbb{R}^2$

"Stokes Thm" higher dimensions.

$$\underbrace{\oint_{\partial R} V \cdot ds}_{\text{line integral}} = \underbrace{\iint_R (\text{curl } V) \cdot \hat{n}\, dA}_{\text{surface integral}}$$

$\hat{n}$ ↖ unit normal vector to surface R.

Examples.

Calculate $\oint_{\substack{\text{unit circle} \\ \text{(CCW)}}} x\,dy + (x^2-y)\,dx$.

two ways.

Note: Same as $V = (x^2-y,\, x)$

$$\oint_{\text{circle}} V \cdot ds$$

Solution: Is it conservative? optional.

$$d\left(x\,dy + (x^2-y)\,dx\right)$$

$$dx \wedge dy + (2x\,dx - dy) \wedge dx$$

$$= dx \wedge dy - dy \wedge dx$$

$$= 2\,dx \wedge dy. \quad \text{(Not conservative)}$$

unit circle

$$\alpha(t) = (\cos(t), \sin(t)) \qquad 0 \le t \le 2\pi$$

$$V = (x^2-y,\, x)$$

$$\oint V \cdot ds = \int_0^{2\pi} \left(\cos^2(t) - \sin(t), \cos(t)\right) \cdot \underbrace{(-\sin(t), \cos(t))\,dt}_{\alpha'(t)\,dt}$$

$$\oint V \cdot ds = \int_0^{2\pi} -\sin(t)\cos^2(t) + \underbrace{\sin^2(t) + \cos^2(t)}_{1} \, dt$$

$$= \int_0^{2\pi} -\sin(t)\cos(t)\,dt + \int_0^{2\pi} 1 \, dt$$

$u = \sin(t)$
$du = \cos(t)\,dt$

$$= \underbrace{\int_0^0 -u \, du}_{0} + 2\pi = \boxed{2\pi}$$

Using Stoke's (Green's Thm).

$$\int_{\text{unit circle}} x\,dy + (x^2 - y)\,dx = \iint_{\text{disk}} d\left(x\,dy + (x^2-y)\,dx\right)$$

$$= \iint_{\text{disk}} \left(dx \wedge dy + (2x\,dx - dy) \wedge dx\right)$$

$$= \iint_{\text{disk}} 2\,dx \wedge dy = 2 \cdot \text{area of disk}$$

$$= \boxed{2\pi}.$$

In 2-d case

$$\oint \vec{v}\cdot d\vec{s} = \oint_{\alpha \text{ ccw}} V_1\,dx + V_2\,dy$$

$$= \iint_{\substack{\text{inside} \\ \text{of } \alpha}} \left(-(V_1)_y + (V_2)_x\right) dx\,dy$$

We had $\vec{v} = (x^2 - y, x)$

$$\oint \vec{v}\cdot d\vec{s} = \int_{\circlearrowleft} (x^2-y)\,dx + x\,dy$$

$$= \iint \left(-(x^2-y)_y + (x)_x\right) dx\,dy$$

$$= \iint (1 + 1)\,dx\,dy$$

$$= 2\,(\text{area of disk})$$

$$= \boxed{2\pi}.$$